UNIFORM HYPERBOLICITY REVISITED: INDEX OF PERIODIC POINTS AND EQUIDIMENSIONAL CYCLES

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ABSTRACT. In this paper we revisit uniformly hyperbolic basic sets and the domination of Oseledets splittings at periodic points. We prove that periodic points with simple Lyapunov spectrum are dense in non-trivial basic pieces of C^r-residual diffeomorphisms on three-dimensional manifolds ($r \geq 1$). In the case of the C^1-topology we can prove that either all periodic points of a hyperbolic basic piece for a diffeomorphism f have simple spectrum C^1-robustly (in which case f has a finest dominated splitting into one-dimensional sub-bundles and all Lyapunov exponent functions of f are continuous in the weak*-topology) or it can be C^1-approximated by an equidimensional cycle associated to periodic points with robust different signatures. The later can be used as a mechanism to guarantee the coexistence of infinitely many periodic points with different signatures.

1. INTRODUCTION

A global view of dynamical systems has been one of the leading problems considered by the dynamical systems community. Based on the pioneering works of Peixoto and Smale, a conjecture proposed by Palis in the nineties has constituted a route guide for a global description of the space of dynamical systems. This program, that roughly describes complement of uniform hyperbolicity as the space of diffeomorphisms that are approximated by those exhibiting either homoclinic tangencies or heteroclinic cycles, has been completed with much success in the C^1-topology, where perturbation tools like the Pugh closing lemma, Franks' lemma, Hayashi's connecting lemma or Mañé ergodic closing lemma developed for the characterization of structural stability are available (see e.g. [9, 8, 12] and references therein).

Although the uniform geometric structures of invariant manifolds present in hyperbolic diffeomorphisms are a basic ingredient in achieving several core results and are quite well established, some important questions on the regularity of finer dynamical properties still remain to be answer.

Motivated by the analysis of the regularity of the Lyapunov exponents similarly to the proof of the stability conjecture, periodic orbits and their eigenvalues should play a key role. We say that a periodic point p of period $\pi(p) \geq 1$ for a diffeomorphism f has *simple spectrum* if all the eigenvalues of $Df^{\pi(p)}(p)$ are real and distinct. Since all periodic points for an Axiom A diffeomorphism, in the same basic piece, have the same index and are homoclinically related, then we will attribute them a signature, which consists of an ordered list of the dimensions of their

finest dominated splittings. Moreover, we will say that two periodic points p, q have different signatures if the Oseledets splittings for $Df^{\pi(p)}(p)$ and $Df^{\pi(q)}(q)$, which consist of the finest dominated splittings at the points, are distinct. Our purpose in this paper is to revisit uniformly hyperbolic basic and explore the notion of domination of Oseledets splittings and characterize the continuity points of Lyapunov exponent functions in this context. These results lie ultimately in the analysis of the eigenvalues and the finest dominated splitting over the periodic points.

In the first part of the paper (see Theorem 1) we prove that periodic points with simple Lyapunov spectrum are dense in non-trivial basic pieces of C^r-residual diffeomorphisms on three-dimensional manifolds ($r \geq 1$). In particular, any diffeomorphism with a hyperbolic basic piece can be C^r-approximated by a diffeomorphism with a dense set of periodic points in the continuation of the hyperbolic set. The proof of this result explores the hyperbolic structure of hyperbolic basic pieces, hence it does not rely on the classical C^1-perturbation lemmas.

In the case of the C^1-topology, one can describe the mechanisms to deduce simple Lyapunov spectrum for all invariant measures assuming the same property at periodic points, in the spirit of the previous works [3, 6, 7, 14]. On the one hand we prove that, if all periodic points of a hyperbolic basic piece for a diffeomorphism f have simple spectrum and the same property holds in a C^1 neighborhood of f (c.f. Definition 2.3) then f has a finest dominated splitting into one-dimensional sub-bundles (see Theorem 2). In this case all Lyapunov exponent functions of f are continuous in the weak*-topology. On the other hand, if periodic points miss to have simple spectrum robustly, then, by a C^1-arbitrarily small perturbation, one can create an equidimensional cycle associated to a pair of periodic points which robustly exhibit different signatures (see Theorem 3). Finally, we justify that these equidimensional cycles associated to periodic points with different signatures play a similar role to the one of tangencies and heterodimensional cycles by its instability character. In fact, the existence of homoclinic tangencies or heterodimensional cycles is often associated to the so-called Newhouse phenomenon of persistence of infinitely many sources or sinks (see e.g. [10] and references therein). In this setting, the existence of a equidimensional cycle associated to periodic points with different signatures can be used to generate (by perturbation) infinitely many periodic points with any of the signatures of the generating periodic orbits (see Theorem 4). Using the existence of Markov partitions we deduce that similar results hold in the time-continuous setting (see Corollary 2).

2. Preliminaries and statement of the main results

2.1. Hyperbolic, Oseledets and finest dominated splittings.
Let M be a compact Riemannian manifold M and $f \in \mathrm{Diff}^1(M)$. Let $\mathrm{Per}(f)$ denote the set of periodic points for f and $\Omega(f) \subset M$ denote the non-wandering set of f. Given an f-invariant compact set $\Lambda \subseteq M$ we say that Λ is a *uniformly hyperbolic set* if there is a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0, 1)$ so that

$$\|Df^n(x)\mid_{E_x^s}\| \leq C\lambda^n \quad \text{and} \quad \|(Df^n(x)\mid_{E_x^u})^{-1}\| \leq C\lambda^n$$

for every $x \in \Lambda$ and $n \geq 1$. We can always change the metric in order to obtain $C = 1$. We refer to $T_\Lambda M = E^s \oplus E^u$ as the *hyperbolic splitting* associated to f. We say that Λ is an *isolated set* if there exists an open neighborhood $U \supset \Lambda$ such that $\Lambda = \cap_{n \in \mathbb{N}} f^n(U)$. Finally, we say that Λ is *transitive* if there exists $x \in \Lambda$ such

that the f-orbit of x is dense in Λ. Along the paper unless stated otherwise we always assume that Λ is a uniformly hyperbolic, isolated and transitive set. For simplicity we call such a set a *hyperbolic basic piece* or simply a *basic piece*. It is well-know that hyperbolic sets admit analytic continuations, that is, there exists a C^1-neighborhood $\mathcal{U}$ of f and an open neighborhood U of Λ so that $\Lambda_g := \cap_{n \in \mathbb{N}} g^n(U)$ is a basic piece for g and $f|_\Lambda$ is topologically conjugated to $g|_{\Lambda_g}$. Finally, we say that a hyperbolic basic piece is *non-trivial* if it does not consist of a hyperbolic periodic point.

We say that a C^1-diffeomorphism f is *Axiom A* if $\overline{\mathrm{Per}(f)} = \Omega(f)$ and $\Omega(f)$ is a uniformly hyperbolic set. We observe that the non-wandering set of an Axiom A diffeomorphism can be decomposed in a finite number of basic pieces. If μ is an f-invariant probability measure, then it follows from Oseledets' theorem [11] that for μ-almost every x there exists a decomposition (*Oseledets splitting*) $T_x M = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^{k(x)}$ and, for $1 \le i \le k(x)$, there are well defined real numbers

$$\lambda_i(f, x) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x)v_i\|, \quad \forall v_i \in E_x^i \setminus \{\vec{0}\}$$

called the *Lyapunov exponents* associated to f and x. It is well known that, if μ is ergodic, then the Lyapunov exponents are almost everywhere constant and $k(x) = k$ is constant. In this case the Lyapunov exponents are denoted simply by $\lambda_i(f, \mu)$. The *Lyapunov spectrum* of a probability measure is the collection of all its Lyapunov exponents. If Λ is a hyperbolic set for f and μ is an f-invariant and ergodic probability measure supported on Λ, Poincaré recurrence theorem implies that $\mathrm{supp}(\mu) \subset \Omega(f)$ and, consequently, the decomposition $T_x M = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^k$ is finer than $T_x M = E_x^s \oplus E_x^u$. In particular μ is hyperbolic i.e. has only non-zero Lyapunov exponents.

Definition 2.1. *Given a Df-invariant decomposition $F = E^1 \oplus E^2 \subset TM$ over a basic piece Λ of a diffeomorphism f we say E^1 is dominated by E^2 if there exists $C > 0$ and $\lambda \in (0, 1)$ so that*

$$\|Df^n(x)\mid_{E_x^1}\| . \|(Df^n(x)\mid_{E_x^2})^{-1}\| \le C\lambda^n \text{ for every } n \ge 1 \text{ and } x \in \Lambda.$$

It is known that we can change the metric in order to obtain $C = 1$. It is clear from the definition that any hyperbolic splitting is necessarily a dominated splitting.

Definition 2.2. *We say that f has a one-dimensional finest dominated splitting over a basic piece Λ if there exists a continuous Df-invariant decomposition $T_x M = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^{\dim M}$ in one-dimensional subspaces at every $x \in \Lambda$ such that E_x^i is dominated by E_x^{i+1} for every $i = 1 \ldots \dim M - 1$.*

It is clear from the previous definition that having a one-dimensional finest dominated splitting is a C^1-open condition for C^r-diffeomorphisms ($r \ge 1$). Since dominated splittings vary continuously with the base point on a basic set Λ, if the diffeomorphism f has a one-dimensional finest dominated splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^{\dim M}$ then there exists a continuous function

$$\Lambda \ni x \mapsto (v_x^1, v_x^2, \ldots, v_x^{\dim M}) \in (T_x^1 M)^{\dim M}$$

such that $v_x^i \in E_x^i$ for every $1 \le i \le \dim M$ and $x \in \Lambda$ and, consequently, the limit superior

$$\overline{\lambda}_i(f, x) := \limsup_{n \to \infty} \frac{1}{n} \log \|Df^n(x)v_i\| \tag{2.1}$$

is everywhere well defined and it does not depend on the choice of the vector $v_i \in E_x^i \setminus \{0\}$. Defining $S : T_\Lambda^1 M \to T_\Lambda^1 M$ and $\phi : T_\Lambda^1 M \to \mathbb{R}$ by

$$S(x, v) = \left(f(x), \frac{Df(x)v}{\|Df(x)v\|} \right) \quad \text{and} \quad \phi(x, v) = \log \|Df(x)v\|$$

then it follows that each value $\overline{\lambda}_i(f, x)$ can be written as

$$\overline{\lambda}_i(f, x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(S^j(x, v_i^x)). \tag{2.2}$$

Using (2.3), for any f-invariant and ergodic probability measure μ each of the values $\overline{\lambda}_i(f, x)$ coincide μ-almost everywhere coincide with the Lyapunov exponents $\lambda_i(f, \mu)$, for every $1 \le i \le \dim M$. More generally, for any f-invariant probability measure μ we define their *Lyapunov exponents* by

$$\lambda_i(f, \mu) := \int \overline{\lambda}_i(f, x) \, d\mu(x) \tag{2.3}$$

for every $1 \le i \le \dim M$.

Remark 2.1. *If μ is an f-invariant probability measure, the ergodic decomposition theorem guarantees that $\mu = \int \mu_x \, d\mu(x)$ where each μ_x is an f-invariant and ergodic probability measure defined for μ-almost every $x \in \Lambda$, and $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ converges to μ_x in the weak*-topology. Thus for μ-almost every $x \in \Lambda$ and every $1 \le i \le \dim M$,*

$$\overline{\lambda}_i(f, x) = \lambda_i(f, \mu_x) = \int \phi(y, v_i^y) \, d\mu_x(y).$$

Equality in (2.3) can also be written as

$$\lambda_i(f, \mu) = \int \overline{\lambda}_i(f, x) \, d\mu(x) = \int \lambda_i(f, \mu_x) \, d\mu(x) = \int \phi(x, v_i^x) \, d\mu(x) \tag{2.4}$$

for every $1 \le i \le \dim M$.

2.2. **Statement of the main results.**

2.2.1. *Plenty of periodic orbits with simple spectrum on hyperbolic basic sets of C^r-diffeomorphisms.* In this subsection we discuss the simplicity of the Lyapunov spectrum for C^r-diffeomorphisms $(r \ge 1)$ restricted to hyperbolic basic pieces.

Theorem 1. *Assume $\dim M = 3$ and $r \ge 1$. Let f be a C^r-diffeomorphism and Λ be a non-trivial hyperbolic basic set for f and let $\mathcal{U}$ be a C^r-open neighborhood of f so that the hyperbolic continuation Λ_g is well defined for all $g \in \mathcal{U}$. Then, there exists a C^r-residual subset $\mathcal{R} \subset \mathcal{U}$ such that if $g \in \mathcal{R}$ then the set of periodic orbits with simple spectrum is dense in Λ_g.*

Some comments are in order. In the case that $r = 1$ there are some perturbation tools sufficient to guarantee the previous result (e.g. the methods of [3] used in the proof of Theorem 3). In the case that $r > 1$, where there is a lack of perturbation

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tools, we use strongly both the robustness of the hyperbolic splitting and the persistence of the hyperbolic basic piece to prove that generic diffeomorphisms have a dense set of periodic orbits with simple spectrum on non-trivial basic pieces. We also observe that the three-dimensional assumption on the manifold is crucial.

2.2.2. *The simple star property and one-dimensional finest dominated splittings.* In this subsection we provide a criterium over the periodic points for a hyperbolic basic piece to admit a one-dimensional finest dominated splitting.

Definition 2.3. *Given a basic piece Λ for a diffeomorphism f we say that $f|_\Lambda$ is simple star if there exists a C^1-open neighborhood $\mathcal{U}$ of f so that, for any $g \in \mathcal{U}$ all periodic points for $g|_{\Lambda_g}$ have simple spectrum.*

We observe that after [9], a C^1-diffeomorphism is called star if it admits a C^1-open neighborhood formed by diffeomorphisms whose periodic points are all hyperbolic. Since the non-wandering set is uniformly hyperbolic then it is clear that the set of Axiom A diffeomorphisms forms a C^1-open subset of star diffeomorphisms. It follows from [3] that periodic points are enough to determine dominated splittings at the closure of periodic points. Since periodic points are dense in basic pieces we obtain the following result.

Theorem 2. *If $f|_\Lambda$ is simple star, then f admits a one-dimensional finest dominated splitting over Λ. Moreover, the Lyapunov spectrum is simple for all f-invariant ergodic probability measures and the map*

$$
\begin{aligned}
\mathscr{L} : \mathcal{M}_1(f|_\Lambda) &\to && \mathbb{R}^{\dim M} \\
\mu &\mapsto && (\lambda_1(f,\mu),\ldots,\lambda_{\dim M}(f,\mu))
\end{aligned}
\tag{2.5}
$$

is continuous, where $\mathcal{M}_1(f|_\Lambda)$ denotes the space of f-invariant probability measures supported in Λ and endowed with the weak* topology.

We observe that if the diffeomorphism f has a one-dimensional finest dominated splitting over Λ, then every f-invariant probability measure supported on Λ admits $d = \dim M$ distinct Lyapunov exponents that differ by a definite amount. Thus, the following is a direct consequence of Theorem 2.

Corollary 1. *Let Λ be a basic piece of a diffeomorphism f. The following properties are equivalent:*

  (a) *$f|_\Lambda$ is simple star;*
  (b) *$f|_\Lambda$ admits a one-dimensional finest dominated splitting;*
  (c) *there exists a constant $c > 0$ so that any f-invariant probability measure μ supported in Λ has $\dim M$ distinct Lyapunov exponents and $\lambda_{i+1}(f,\mu) - \lambda_i(f,\mu) \geq c > 0$ for every $i = 1,\ldots,\dim M - 1$.*

The later corollary ultimately means that if the Lyapunov spectrum of periodic points simple is C^1-robust, then there exists a uniform gap in the Lyapunov spectrum of periodic points.

2.2.3. *A dichotomy for basic pieces.* In this subsection we obtain a dichotomy of one-dimensional finest dominated splitting *versus* equidimensional cycles with robust different signatures for basic pieces. Let us introduce these notions more precisely. Let Λ be a basic set of a diffeomorphism f and $p \in \Lambda$ be a hyperbolic periodic point of period $\pi(p) \geq 1$. Given the finest $Df^{\pi(p)}(p)$-invariant dominated splitting $E_p^u = E_1^u \oplus \cdots \oplus E_k^u$ we define the *unstable signature at p* (and denote it

by $\mathrm{sgn}^u(p)$) to be the k-uple $(\dim E_1^u, \ldots, \dim E_k^u)$. Stable signatures at periodic points are defined analogously and denoted by sgn^s. Unstable (resp. stable) signatures describe the existence of finer splittings in the unstable (resp. stable) bundle of periodic points.
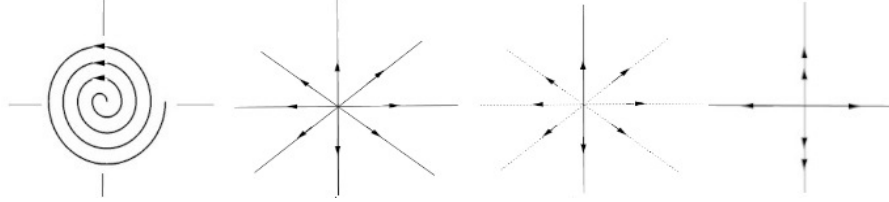

FIGURE 1. Different linear behavior of $Df^{\pi(p)}(p)\mid_{E_p^u}$ in the case of two-dimensional unstable subbundle corresponding to signatures (2), (2), (2) and $(1,1)$, respectively. The third figure from the left corresponds to equal eigenvalues with a non-zero nilpotent part.
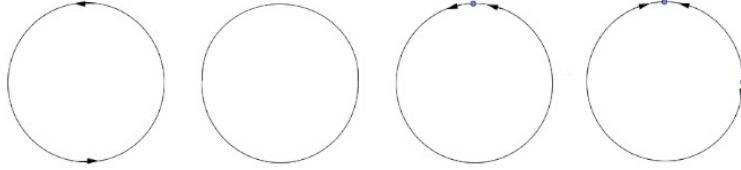

FIGURE 2. Action of $Df^{\pi(p)}(p)\mid_{E_p^u}$ on the projective space in the case of two-dimensional unstable sub-bundle ordered as Figure 1.

Definition 2.4. *We say that a C^1 diffeomorphism f exhibits an* equidimensional cycle with robust different signatures *on a basic piece Λ if there are two periodic points $p, q \in \Lambda$ with different stable or unstable signatures, and this property holds under arbitrarily C^1-small perturbations.*
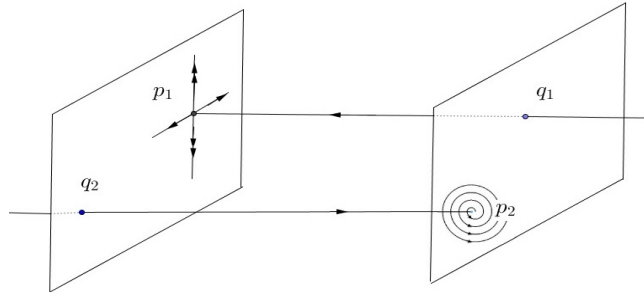

FIGURE 3. Equidimensional cycle between two fixed points p_1 and p_2 such that $\mathrm{sgn}^u(p_1) = (1,1)$ and $\mathrm{sgn}^u(p_2) = (2)$.

In dimension three, the robustness of the different signatures is associated to the existence of a pair of periodic points whose action of the dynamical cocycle on

the projective space of one periodic point exhibits a dominated splitting while for the other point it presents a rotational effect (c.f. Figures 1-3). This is in strong analogy with the lack of domination observed in the presence of homoclinic tangencies. In the space of C^1-diffeomorphisms the presence of homoclinic tangencies and heterodimensional cycles appears as a mechanism to create infinitely many sources or sinks (see e.g. the Newhouse phenomenon in [10]). Roughly, if there exists a homoclinic tangency associated to a dissipative periodic point of saddle type then the lack of domination allows to perturb the system to mix the directions and to create a sink/source. The infinitely many sinks/sources arise associated to some phenomenon of persistence of tangencies.

Theorem 3. *Assume that* $\dim M = 3$. *Let* Λ *be a non-trivial basic piece of a diffeomorphism* f *and* $\mathcal{U}$ *a* C^1*-open neighborhood of* f *on which the analytic continuation* Λ_g *of* Λ *is well-defined for all* $g \in \mathcal{U}$. *Either* $f|_\Lambda$ *has a one-dimensional finest dominated splitting, or* f *can be* C^1*-approximated by diffeomorphisms* g *that exhibit a equidimensional cycle in* Λ_g *with robust different signatures.*

We notice that the property of having one-dimensional finest dominated splitting and the property of exhibiting a equidimensional cycle with robust different signatures are C^1-open. It is not hard to construct examples that lie in the boundary of the diffeomorphisms with a one-dimensional finest dominated splitting in a basic piece but not on the closure of diffeomorphisms exhibiting a equidimensional cycle with robust different signatures, due to the existence of nilpotent part. In particular, there are examples of C^1 diffeomorphisms exhibiting periodic points (of low period) with equal eigenvalues that cannot be perturbed to create a complex eigenvalue. After this work was completed we were informed that, in the C^1-topology, Bochi and Bonatti [1] describe all Lyapunov spectra that can be obtained by perturbing the derivatives along periodic orbits of a diffeomorphism.

The next result explains the conceptual similarity between equidimensional cycles with robust different signatures and the notion of heterodimensional cycles for dynamics beyond uniform hyperbolicity.

Theorem 4. *Assume that* $\dim M = 3$ *and let* Λ *be a non-trivial basic piece of a diffeomorphism* f. *If* $f|_\Lambda$ *admits two periodic points* p, q *with robust different unstable signatures, then there exists a* C^1*-open neighborhood* $\mathcal{U}$ *of* f *and a residual subset* $\mathcal{R} \subset \mathcal{U}$ *such that for any* $g \in \mathcal{R}$ *we have that:*

(1) *there exists a countable infinite set of points* $P_1 \subset Per(g) \cap \Lambda_g$ *with* $sgn^u(p_1) = sgn^u(p)$ *for all* $p_1 \in P_1$ *and*
(2) *there exists a countable infinite set of points* $P_2 \subset Per(g) \cap \Lambda_g$ *with* $sgn^u(p_2) = sgn^u(q)$ *for all* $p_2 \in P_2$,

where Λ_g *stands for the hyperbolic continuation of the set* Λ.

An application for flows. In what follows we apply our results to the case of hyperbolic basic pieces for C^1-flows. By Bowen and Ruelle [5], the restriction of a C^1-flow to a hyperbolic basic piece for a C^1-flow admits a finite Markov partition and is semiconjugate to a suspension flows over a subshift of finite type and there exists a one-to-one correspondance between invariant measures of the discrete-time and continuous-time dynamical systems. Indeed, given a measurable space Σ, a map $R\colon \Sigma \to \Sigma$, an R-invariant probability measure $\tilde{\mu}$ defined in Σ and a ceiling function $h\colon \Sigma \to \mathbb{R}^+$ in $L^1(\tilde{\mu})$ and bounded away from zero, consider the space

$M_h \subseteq \Sigma \times \mathbb{R}_+$ defined by $M_h = \{(x,t) \in \Sigma \times \mathbb{R}_+ : 0 \leq t \leq h(x)\}/\sim$ where $\sim$ stands for the identification between the pairs $(x, h(x))$ and $(R(x), 0)$. The semiflow defined on M_h by $S^s(x, r) = (R^n(x), r + s - \sum_{i=0}^{n-1} h(R^i(x)))$, where $n \in \mathbb{N}_0$ is uniquely defined by $\sum_{i=0}^{n-1} h(R^i(x)) \leq r + s < \sum_{i=0}^{n} h(R^i(x))$ is called a *suspension semiflow*. If R is invertible then $(S^t)_t$ is indeed a flow. Since h is bounded below, if Leb_1 denotes the one dimensional Lebesgue measure, then $\eta \mapsto \bar{\eta} := \frac{\eta \times \mathrm{Leb}_1}{\int h \, d\eta}$ is a one-to-one correspondence between R-invariant probability measures and S^t-invariant probability measures (we refer the reader [5] for more details).

Hence, if $T_x\Sigma = E_x^1 \oplus \ldots E_x^k$ for a η-generic point x then $T_{(x,s)}\Sigma = E_{(x,s)}^1 \oplus \ldots E_{(x,s)}^k \oplus \langle X \rangle$ is the Oseledets splitting associated to the vector field X for $\bar{\eta}$-almost every (x, s), where $\langle X \rangle$ is the subbundle generated by the vector field. In particular, the Lyapunov spectrum of $(X, \bar{\eta})$ is given by the Lyapunov exponents of (S, η) and zero (corresponding to the flow direction). Given the previous characterization, a hyperbolic basic piece Λ for a vector field X, the vector field $X \mid_\Lambda$ is C^1-simple star if and only if the Poincaré map $S \mid_{\Sigma \cap \Lambda}$ is C^1-simple star. We also stress that such a Poincaré map inherits the information of the Lyapunov exponents for both periodic points and all invariant measures: the Lyapunov exponents for the flow and for the Poincaré map differ from the integral of the return time function. The definitions from the discrete time setting extend to this setting in an analogous way. Since the C^1-perturbation tools used in the proofs of Theorems 2 and 3 are available for flows, then the following result is a consequence of the above mentioned theorems.

Corollary 2. *If $(X_t)_t$ is a simple star structurally stable flow, then $(X_t)_t$ admits a one-dimensional finest dominated splitting and the Lyapunov exponent functions vary continuously with the invariant probability measure. Moreover, if $\dim M = 4$ we have the following dichotomy: if $(X_t)_t$ is a structurally stable flow then either $(X_t)_t$ admits a one-dimensional finest dominated splitting, or $(X_t)_t$ can be C^1-approximated by flows that exhibit a equidimensional cycle associated points with robust different signatures.*

3. Proofs

In this section we prove our main results.

3.1. **Proof of Theorem 1.** The arguments here use strongly that M is a three-dimensional manifold. Let $r \geq 1$, let f be a C^r-diffeomorphism and Λ be a non-trivial hyperbolic basic set for f and let $\mathcal{U}$ be a C^r-open neighborhood of f so that the hyperbolic continuation Λ_g is well defined for all $g \in \mathcal{U}$.

In particular, for every $g \in \mathcal{U}$ there exists an homeomorphism $h_g : \Lambda \to \Lambda_g$ close to the identity so that $h_g \circ f \mid_\Lambda = g \mid_{\Lambda_g} \circ h_g$. Consequently, if $(p_j)_{j=1\ldots\ell}$ are the periodic points of period m for $f \mid_\Lambda$ then the continuations $(h_g(p_j))_{j=1\ldots\ell}$ are the periodic points of period m for g. Let $Per_s(g)$ denote the set of periodic points with simple Lyapunov spectrum. The set

$$\mathcal{U}_n = \left\{ g \in \mathcal{U} \colon Per_s(g) \text{ is } \frac{1}{n}\text{-dense in } \Lambda_g \right\}$$

is clearly a C^r-open subset of $\mathcal{U}$. We claim that $\mathcal{U}_n$ is a C^r-dense subset of $\mathcal{U}$. This will be enough to prove the theorem because the residual subset $\mathcal{R} := \bigcap_{n\in\mathbb{N}} \mathcal{U}_n \subset \mathcal{U}$ is so that the set of periodic points with simple Lyapunov spectrum for any $g \in \mathcal{R}$ is dense in Λ_g. In other words, the residual subset $\mathcal{R} \subset \mathcal{U}$ satisfies the requirements

of the theorem. Thus, we are left to prove that for every $n \geq 1$ the set $\mathcal{U}_n$ is a C^r-dense subset of $\mathcal{U}$.

Before that we need a simple and useful lemma to perform the local C^r-perturbations. This lemma is quite different from the well-known Franks' lemma because the approximation on the derivative depends on a pre-fixed size of neighborhood of the support of the perturbation.

Lemma 3.1. *Given a C^r diffeomorphism $f \colon M \to M$, $\epsilon > 0$, $p \in M$ and $R > 0$, there exists $\delta := \delta(\epsilon, R) > 0$ such that the following holds: if $A(p) \colon T_x M \to T_{f(x)} M$ is δ-close to $Df(p)$ in the uniform norm of operators then there exists a C^r diffeomorphism g such that:*

   (1) *g is ϵ-C^r-close to f;*
   (2) *g is supported in $B(p, R)$ and*
   (3) *$Dg(p) = A(p)$.*

Proof. For simplicity we assume all the computations in local charts in $\mathbb{R}^n$ and that p is fixed. Take $\epsilon > 0$ and $R > 0$ and consider a C^∞ *bump function* $\varphi \colon [0, +\infty[\to [0, 1[$ such that $\varphi(t) = 0$ if $t > \sqrt{R}$ and $\varphi(t) = 1$ if $t < \frac{\sqrt{R}}{2}$. It follows from Faà di Bruno's formula that, given $q \in B(p, R)$, we have

$$\frac{\partial^n \varphi(\|q\|^2)}{\partial q_k} = \sum \frac{n!}{m_1! 1!^{m_1} m_2! 2!^{m_2} \ldots m_n! n!^{m_n}} \varphi^{(m_1 + \ldots + m_n)}(\|q\|^2) \prod_{j=1}^n \left(\frac{\partial^j \|q\|^2}{\partial q_k} \right)^{m_j},$$

(3.1)

where the sum is over all vectors with nonnegative integers entries $(m_1, \ldots, m_n)$ such that we have $\sum_{j=1}^n j.m_j = n$. These derivatives are bounded above by a constant C (depending on R).

Let $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map δ-close to $Df(p)$, where $\delta > 0$ is very small (to be determined later on and depending on the derivatives in (3.1)). Take a C^r smooth map $\hat{h} \colon B(p, R) \to \mathbb{R}^n$ defined by $\hat{h}(q) = [Df(p)]^{-1} \cdot A(p)$, take $h(q) = \varphi(\|q\|^2)\hat{h}(q) + (1 - \varphi(\|q\|^2))Id$ and define $g := f \circ h$. Clearly, by construction, g coincides with f in the complement of $B(p, R)$ and

$$Dg(p) = Df(h(p)) \cdot Dh(p) = Df(p) \cdot D\hat{h}(p) = Df(p) \cdot [Df(p)]^{-1} \cdot A(p) = A(p).$$

This proves (2) and (3). Furthermore, since $\hat{h}$ is constant and $\|\hat{h}(q)\| < \delta$, then the C^r-distance between f and g is proportional to $\|\hat{h}\|$ and can be taken smaller than ϵ provided that δ is small enough. $\square$

We are now in a position to proceed with the proof that for every $n \geq 1$ the set $\mathcal{U}_n$ is a C^r-dense subset of $\mathcal{U}$. In fact this is a consequence of the following

Proposition 3.1. *Given $\epsilon > 0$, $n \geq 1$ and a non-trivial basic piece Λ of a C^r-diffeomorphism f, there exists a C^r-diffeomorphism g that is ϵ-C^r-close to f and displaying a g-periodic orbit q that is $\frac{1}{n}$-dense in Λ_g and has simple Lyapunov spectrum.*

Proof. Let $\epsilon > 0$, f be a C^r diffeomorphism, and Λ be a non-trivial basic piece for f. Assume without loss of generality that $T_\Lambda M = E^s \oplus E^u$ with $\dim E^s = 1$ and $\dim E^u = 2$ (otherwise just consider the diffeomorphism f^{-1}). In this three-dimensional setting, $p \in \text{Per}(f)$ has simple Lyapunov spectrum if and only if there are no complex eigenvalues for $Df^{\pi(p)}(p) \mid_{E_p^u}$ or $Df^{\pi(p)}(p) \mid_{E_p^u}$ has no equal real eigenvalues. Notice that this last mentioned situation has empty C^r-interior.

Fix $p \in \mathrm{Per}(f)$. Up to consider $f^{\pi(p)}$ we just assume that $\pi(p) = 1$ and that the restriction of f to the non-trivial hyperbolic homoclinic class Λ associated to p is topologically mixing. In particular, $f \mid_\Lambda$ satisfies the specification property: for each $\zeta > 0$, there is an integer $N(\zeta)$ for which the following is true: if $I_1, I_2, \cdots, I_k$ are pairwise disjoint intervals of integers with

$$\min\{|m - n| : m \in I_i, n \in I_j\} \geq N(\zeta)$$

for $i \neq j$ and $x_1, \cdots, x_k \in X$ then there is a point $x \in X$ such that $d(f^j(x), f^j(x_i)) \leq \zeta$ for $j \in I_i$ and $1 \leq i \leq k$. Every topologically mixing hyperbolic elementary set of a diffeomorphism satisfies the specification property (see [4]).

Consider a $\frac{1}{2n}$-dense set F in Λ. Then there exists $\ell = \ell(n) \geq 1$ so that for any large m there exists a periodic point $x_m \in \Lambda$ of period equal to $m + \ell \times \#F$, such that the first m iterates of x_m belong to a small neighborhood U_0 of p and the piece of orbit $\{f^{j+m}(x_m) : 1 \leq j \leq \ell\}$ is $\frac{1}{n}$-dense.

Observe that ℓ is fixed and depends only on n. Defined in this way, as m increases, the point x_m is sufficiently close to p and spends a large amount of time near p, and the matrix $Df^m(x_m)$ inherits the dynamical behavior of $[Df(p)]^m$ whereas the matrix $Df^{m+\ell}(x_m)$, corresponding to the periodic orbit computed along the whole orbit of x_m. Define the compact f-invariant hyperbolic sets

$$K_n(f) := \overline{\bigcup_{m \geq n} \bigcup_{i \in \mathbb{Z}} f^i(x_m)} \subset \Lambda, \quad n \geq 1. \tag{3.2}$$

The persistence of the hyperbolic splitting $E_f^u \oplus E_f^s$ for the map f under C^1 (thus C^r) perturbations allows us to conclude that any g sufficiently C^r-close to f also has a hyperbolic splitting $E_g^u \oplus E_g^s$ (with a two-dimensional unstable and one-dimensional stable bundles E_g^u and E_g^s, respectively) over the g-invariant compact set $K_n(g)$ obtained as in (3.2) by taking the continuation of the hyperbolic periodic points $(x_m)_m$.

Now we will show that some C^r-small perturbation g can be made in order to find a periodic point x_m in $K_n(g)$ such that all the eigenvalues of $Dg^{\pi(x_m)}(x_m)$ are real and distinct, where $\pi(x_m) = m + \ell \times \#F$. Such point x_m is $\frac{1}{n}$-dense by construction. Actually, although explicitly defined at a neighborhood of p, the detailed construction of the perturbation map necessarily makes use of bump-functions. For shortness we shall omit the full detailed construction. For any $n \geq 1$, let $\rho(m, f)$ stand for the rotation number of $Df^{\pi(x_m)}(x_m) \mid_{E_{x_m}^u}$. Given a continuous arc of maps $(f_t)_{t \in I}$ near f let $\delta(m, f_t)$ denote the oscillation of the rotation number along the arc, that is, $\delta(m, (f_t)_t) = \sup\{|\rho(m, f_t) - \rho(m, f_s)| : s, t \in I\}$. The next claim corresponds to [2, Lemma 9.3] in our setting.

Claim: There exists a continuous arc of maps $\{f_t\}_{t \in [0,1]}$ satisfying $f_0 = f$ and $\|f_t - f_0\|_{C^r} < \epsilon$ for all $t \in [0, 1]$ so that the following holds: for every t there exists $m_t \in \mathbb{N}$ so that $\delta(m, (f_s)_{s \in [0,t]}) > 1$ for all $m \geq m_t$.

Proof of the claim: We consider from now on a basis adapted to the splitting $E_p^u \oplus E_p^s$ (for f), a constant $\sigma \in (0, 1)$, $\gamma > 1$ and maps $h_{t\xi}$ with support in a small ball $B(p, R) \subset U_0$ such that:

$$Df(p) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \gamma \cos\theta & -\gamma \sin\theta \\ 0 & \gamma \sin\theta & \gamma \cos\theta \end{pmatrix} \quad \text{and} \quad Dh_{t\xi}(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t\xi) & -\sin(t\xi) \\ 0 & \sin(t\xi) & \cos(t\xi) \end{pmatrix}$$

As in Lemma 3.1, consider the C^r-diffeomorphisms $f_t = h_{t\xi}(p) \circ f(p)$ and let ξ be sufficiently small in order to have f_t ϵ-C^r-close to f for all $t \in [0, 1]$. Since the uniformly hyperbolic splitting varies continuously with the point and the angle between stable and unstable bundles is bounded away from zero each fiber E_x^u and E_x^s is a graph over E_p^u and E_p^s (for all maps in the family $(f_t)_{t\in[0,1]}$) for all points x in the hyperbolic isolated set. Identifying all the stable and unstable subspaces with E_p^s and E_p^u by parallel transport, consider the decomposition

$$Df_t^{\pi(x_m)}(x_m) = \alpha_{t,m,m}.(\dots).\alpha_{t,m,1}.\beta_{t,m},$$

where the α's correspond to the derivative Df_t at iterates of x_m by f_t inside the neighborhood U_0 of p and the β's to the derivative Df_t at the iterates of x_m by f_t that are outside it. Since, for each m we only allow $\ell(n) \times \#F$ iterates outside the neighborhood U_0 we obtain that, up to consider some subsequence if necessary, $\beta_{t,m}$ converges uniformly to some β_t as $n \to \infty$ (i.e. the period of the x_m's increases). We need to prove that for any t there exists $m_t \in \mathbb{N}$ so that $\delta(m, (f_s)_{s\in[0,t]}) > 1$ for all $m \geq m_t$. Notice that each $\alpha_{t,m,i} |_{E^u}$ is uniformly close to the composition of an homothety by a rotation of angle $t\xi + \theta$ and so all give a contribution to the increase of the original rotation number by some positive constant and the claim is proved. $\qquad\square$

By the claim, for any $\epsilon > 0$ there exists a small t and a large m such that f_t is ϵ-C^r-close to f and $\rho(m, f_t) \in \mathbb{Z}$. Therefore, $Df_t^{\pi(x_m)}(x_m)|_{E_{f_t}^u}$ has two equal real eigenvalues of norm larger than 1. Then, we perturb once more to obtain simple spectrum by simply consider a 'diagonal' perturbation at x_m given for $\eta_1, \eta_2 > 0$ arbitrarily small and $\eta_1 \neq \eta_2$, in a suitable base of eigenvalues, by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}.$$

The argument to realize such perturbation of the derivative by a C^r-close diffeomoprhism follows the lines of Lemma 3.1. This completes the proof of the proposition. $\qquad\square$

3.2. **Proof of Theorem 2.** Assume that $f|_\Lambda$ is simple star. We explore the fact that the set of (hyperbolic) periodic points is dense in Λ. If Λ consists of a hyperbolic periodic point the result is immediate. So, we assume that Λ is a non-trivial hyperbolic set.

Using that a dominated splitting extends to the closure of a set and that $f|_\Lambda$ is simple star (thus cannot be C^1-approximated by a diffeomorphism so that the Lyapunov spectrum at some periodic point is not simple) then, by Corollary 3.1, there exist $m_1, n_1 \geq 1$ so that there exists an m_1-dominated splitting $E \oplus F$ over the set $\Lambda \cap \overline{\bigcup_{\ell \geq n_1} Per_\ell(f)}$.

If E has dimension one we take $E^1 = E$. Otherwise, repeating the procedure with $Df|_E$ there exists $n_2 \geq 1$ and $m_2 \geq 1$ (multiple of m_1) and a m_2-dominated splitting $T_{\Lambda \cap \overline{\bigcup_{\ell \geq n_2} Per_\ell(f)}} M = (\hat{E}^1 \oplus \hat{E}^2) \oplus F$ with $0 < \dim \hat{E}^1, \dim \hat{E}^2 < \dim E$. If $\hat{E}^1$ has dimension one we take $E^1 = \hat{E}^1$. Proceeding recursively with E and F it follows that there exist $m \geq 1$ and an m-dominated splitting

$$T_{\Lambda \cap \overline{\bigcup_{\ell \geq n} Per_\ell(f)}} M = E^1 \oplus \cdots \oplus E^{\dim M}.$$

Using that Λ is a (non-trivial) homoclinic class and that every non-trivial homoclinic class contains infinitely many periodic points then $\bigcup_{\ell \geq n} Per_\ell(f)$ is dense in Λ. Thus $f\mid_\Lambda$ has a one-dimensional finest dominated splitting, and the first claim in the theorem follows.

Now, we shall prove that the Lyapunov exponent function associated to $f\mid_\Lambda$ is continuous. In fact, let $T_\Lambda M = E^1 \oplus \cdots \oplus E^{\dim M}$ be the one-dimensional, Df-invariant finest dominated splitting associated to f. Since dominated splittings varies continuously with the point then there exists a continuous function

$$\Lambda \ni x \mapsto (v_x^1, v_x^2, \ldots, v_x^{\dim M}) \in (T_x^1 M)^{\dim M}$$

such that $v_x^i \in E_x^i$ and, by (2.4), the Lyapunov exponents are given by the integrals of continuous functions $\lambda_i(f, \mu) = \int \phi(x, v_i^x) \, d\mu(x)$ for every $1 \leq i \leq \dim M$ and $x \in \Lambda$.

It is immediate that these vary continuously with respect to μ in the weak* topology and, consequently, the Lyapunov exponent map (2.5) is continuous. By the dominated splitting property it follows that the Lyapunov spectrum of every invariant measure μ is simple. This finishes the proof of Theorem 2.

Remark 3.1. *We observe that an alternative argument for the simplicity of the Lyapunov spectrum for all invariant measures follows from the denseness of periodic measures in the set of f-invariant probability measures, by specification* [13].

Remark 3.2. *Since dominated splittings also vary continuously with the diffeomorphism, then every g that is C^1-close to f, then Λ_g is a basic piece with a one-dimensional finest dominated splitting and there exists a continuous function*

$$(x, g) \mapsto (v_{x,g}^1, v_{x,g}^2, \ldots, v_{x,g}^{\dim M}) \in (T_x^1 M)^{\dim M}$$

such that $v_x^i \in E_x^i$. Since Lyapunov exponents are computed as integrals of continuous functions (cf. (2.4)) then given a sequence $(f_n)_n$ convergent to f in the C^1-topology, and invariant measures $\mu_n \in \mathcal{M}_1(f_n)$ that converge to $\mu \in \mathcal{M}(f)$ then we also obtain that $\lim_{n \to \infty} \lambda_i(f_n, \mu_n) = \lambda_i(f, \mu)$ for every $1 \leq i \leq \dim M$.

3.3. **Proof of Theorem 3.** Assume that $\dim M = 3$, Λ is a non-trivial basic piece of a diffeomorphism f and $\mathcal{U}$ a C^1-open neighborhood of f on which the analytic continuation Λ_g of Λ is well-defined for all $g \in \mathcal{U}$. To prove the theorem we are reduced to prove that if $f\mid_\Lambda$ cannot be C^1-approximated by diffeomophisms g so that $g\mid_{\Lambda_g}$ has a one-dimensional finest dominated splitting then f can be C^1-approximated by diffeomorphisms g that exhibits a equidimensional cycle in Λ_g with robust different signatures. Assume that $T_\Lambda M = E^s \oplus E^u$ with $\dim E^s = 1$ and $\dim E^u = 2$. We use the following direct consequence of [3, Corollary 2.18] for the cocycle $A = Df\mid_{E^u}$ together with Franks' lemma and, henceforth, is specific of the C^1-topology.

Corollary 3.1. [3, Corollary 2.18] *For any $\epsilon > 0$ there are two integers m and n such that, for any periodic point x of period $\pi(x) \geq n$:*

(1) *either $Df\mid_{E^u}$ admits an m-dominated splitting along the orbit of x or else;*
(2) *for any neighborhood U of the orbit of x, there exists an ϵ-perturbation g of f in the C^1-topology, coinciding with f outside U and on the orbit of x, and such that the differential $Dg^{\pi(x)}(x)\mid_{E_x^u}$ has all eigenvalues real and with the same modulus.*

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Since all periodic points are homoclinically related, we are reduced to prove first that there are (up to a C^1-perturbation) two periodic points with robust different signatures. We organize our argument in three cases.

Case 1: $Df^{\pi(p)}(p)\mid_{E_p^1}$ has a complex eigenvalue for every $p \in Per(f)$.

If this is the case then $\text{sgn}^u(p) = (2,0)$ for every $p \in \text{Per}(f)$. By Corollary 3.1, item (2), there is a diffeomorphism g, C^1-arbitrarily close to f, and a periodic point p for which $Dg^{\pi(p)}(p)\mid_{E^1}$ has two real eigenvalues with the same modulus. Since this is a non-generic property, by a C^1-small perturbation $\tilde{g}$ of g, the continuation $p_{\tilde{g}}$ of the periodic point p is so that $D\tilde{g}^{\pi(p_{\tilde{g}})}(p_{\tilde{g}})\mid_{E^1}$ has real and simple spectrum. Since $\tilde{g}$ can be obtained displaying some periodic point q in the basic piece such that $D\tilde{g}^{\pi(q)}(q)\mid_{E_q^1}$ has a complex eigenvalue (hence $\text{sgn}^u(q) = (2,0)$ robustly), then $\tilde{g} \in \mathcal{SS}_c^1$.

Case 2: There are $p, q \in Per(f)$ such that $Df^{\pi(p)}(p)\mid_{E_p^1}$ has a complex eigenvalue and $Df^{\pi(q)}(q)\mid_{E_q^1}$ has two real eigenvalues.

If $Df^{\pi(q)}(q)\mid_{E_q^1}$ has simple real spectrum we are done. Otherwise, since there are periodic points whose derivative at the period has a complex eigenvalue in this case a simple and arbitrary small C^1-perturbation g supported in a neighborhood of q in such a way that $Dg^{\pi(q)}(q)\mid_{E_q^1}$ has real simple spectrum and, consequently, an equidimensional cycle with robust different signatures.

Case 3: for all $p \in Per(f)$ the linear map $Df^{\pi(p)}(p)\mid_{E_p^1}$ has only real eigenvalues.

In this case there are three situations to consider, depending on the number of periodic points with two real equal eigenvalues (presenting a nilpotent part or not).

(a) all periodic points $p \in \text{Per}(f)$ are so that $Df^{\pi(p)}(p)\mid_{E_p^1}$ has real simple spectrum (or, in other words, $\text{sgn}^u(p) = (1,1)$). As $f\mid_\Lambda$ cannot be C^1-approximated by diffeomophisms g so that $g\mid_{\Lambda_g}$ has a one-dimensional finest dominated splitting, then it follows from Corollary 1 that f is not a simple star diffeomorphism. Hence, it can be arbitrarily C^1-approximated by a diffeomorphism g so that either there exists a periodic point p such that $Dg^{\pi(p)}(p)\mid_{E^1}$ has one complex eigenvalue or two real equal eigenvalues. Such perturbation can be performed in order to maintain a periodic point with real simple eigenvalues. Moreover, the period of p can be taken as larger as necessary. We are left to prove that a perturbation can be done in order to obtain complex eigenvalues (along E^1) for some periodic point in the case when we have two real equal eigenvalues for $Dg^{\pi(p)}(p)\mid_{E^1}$. We will deal with this situation in the next two cases (b) and (c).

(b) if there exists a periodic point p so that $\dim \text{Ker}(Df^{\pi(p)}(p)\mid_{E_p^1} -I) = 2$ then $Df^{\pi(p)}(p)\mid_{E_p^1}$ has two real eigenvalues larger than one with no nilpotent part. Hence, by a small C^1-perturbation whose support is contained in a neighborhood of p one can obtain a C^1-diffeomorphism g so that $Dg^{\pi(p)}(p)\mid_{E_p^1}$ has a complex eigenvalue (this can be obtained via Franks' lemma and which can be written locally by $Dg = R_\theta \circ Df$, where R_θ represents the rotation of angle θ on the plane E^1 and the identity in E^2).

(c) if all periodic points p have two real equal eigenvalues and are such that $\dim \text{Ker}(Df^{\pi(p)}(p)\mid_{E_p^1} -I) = 1$ we shall borrow an idea from [2, Section 9] and

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proceed as follows. In this case all of these periodic points have a nilpotent part and, in a suitable coordinate system, can be written by

$$Df^{\pi(p)}(p) \mid_{E_p^1} = \begin{pmatrix} \lambda_p & n(p) \\ 0 & \lambda_p \end{pmatrix}$$

where $n(p) \in \mathbb{R}$ denotes the nilpotent part and $\lambda_p > 1$. Actually, since f is Axiom A then it admits a finite Markov partition $\mathcal{P} = \{P_i\}_i$. Let $n_{i,j} \geq 1$ be given such that $f^{n_{i,j}}(P_i) \cap P_j \neq \emptyset$.

Fix $p \in Per(f)$ as above and assume, for simplicity, that $p \in P_1$ is a fixed point. Given $n \geq 1$, let $x_m \in \Lambda$ be a periodic point so that $f^j(x_m) \in P_1$ for every $1 \leq j \leq n$, $f^{n_{1,2}+n}(x_m) \in P_2$ and $x_m = f^{n_{2,1}+n_{1,2}+n}(x_m) \in P_1$. For any $m \gg 1$ the f-invariant compact set $Y_m = \overline{\bigcup_{n \geq m} \bigcup_{j \in \mathbb{Z}} f^j(x_m)}$ inherits the behavior of $Df(p)$. For periodic points $q \in Y_m$ with period $\pi(q) \geq 1$ one has

$$Df^{\pi(q)}(q) \mid_{E_q^1} \approx \begin{pmatrix} \lambda_p^{\pi(q)-\ell} & N(q) \\ 0 & \lambda_p^{\pi(q)-\ell} \end{pmatrix} \cdot A(q,\ell),$$

where $A(q,\ell)$ is a 2×2 matrix corresponding to the $\ell = n_{2,1} + n_{1,2}$ iterates of transitions between P_1 and P_2, and the nilpotent part $N(q)$ grows linearly with $\pi(q)$. In particular, $N(q)/\lambda_p^{\pi(q)} \to 0$ as $\pi(q) \to \infty$. Given $\varepsilon > 0$ choose $q \in Y_m$ with large period so that $N(q) \leq \lambda_p^{\pi(q)-\ell}\varepsilon$.

First we perform a ε-C^1-small perturbation along a piece of orbit of q so that the resulting diffeomorphism g satisfies, in appropriate coordinates,

$$Dg^{\pi(q)}(q) \mid_{E_q^1} \approx \begin{pmatrix} \lambda_p^{\pi(q)-\ell} & 0 \\ 0 & \lambda_p^{\pi(q)-\ell} \end{pmatrix} \cdot A(q,\ell) = \lambda_p^{\pi(q)-\ell}A(q,\ell).$$

If $A(q,\ell)$ have complex eigenvalues we are done.

Otherwise we notice that $A(q,\ell) = Df^\ell(f^{\pi(q)-\ell}(q))$ and that $q \mapsto A(q,\ell)$ is continuous. There exists an appropriate base, given by the Jordan canonical form, so that the matrix $A(q,\ell)$ can be written of the form

$$A_1 := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad A_2 := \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

for some real numbers a, b (depending on the ℓ iterates

$$\{f^j(f^{\pi(q)-\ell}(q)) \colon j = 0 \ldots \ell - 1\}$$

of the orbit of the periodic point q). By the continuity of the matrix $A(q,\ell)$ with the periodic point q the real numbers a, b can be taken bounded from above and below by uniform constants independent on the periodic point q and period $\pi(q)$. The strategy is to perturb the derivative of the diffeomorphism g along a piece of orbit $\{f^j(q) \colon j = 0 \ldots N\}$ ($N \geq 1$ to be determined below) in such a way that the resulting C^1-diffeomorphism $\tilde{g}$ is so that $D\tilde{g}^{\pi(q)}(q) \mid_{E_q^1}$ has complex eigenvalues.

Given $\varepsilon > 0$ take $N(\varepsilon) \approx \frac{1}{\varepsilon}$ (more precisely, $N(\varepsilon) \geq \frac{1}{1+\varepsilon} \log \left| \frac{b}{a} \right|$ in case of matrix A_1 and $N(\varepsilon) \geq \frac{b}{\varepsilon}$ in case of matrix A_2) and take $q \in Y_m$ with period $\pi(q) \geq N(\varepsilon)+\ell$. By Franks' lemma, in the case of a matrix of the form A_1 then we perturb the C^1-diffeomorphism over the piece of orbit $\{f^j(q) \colon j = 0 \ldots j = 0 \ldots N(\varepsilon) - 1\}$ by concatenations of perturbations of the derivative to get a C^1-diffeomorphism g

whose derivative is obtained from the one of f by perturbations (C^0-close to the identity) of the form

$$\begin{pmatrix} 1 + \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$$

in an appropriate base. For the resulting diffeomorphism we get that

$$Dg^{\pi(q)}(q) \mid_{E_q^1} \approx \lambda_p^{\pi(q)-\ell} \begin{pmatrix} (1+\varepsilon)^{N(\varepsilon)}a & 0 \\ 0 & b \end{pmatrix} \approx \lambda_p^{\pi(q)-\ell} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

In the case of a matrix of the form A_2 we proceed analogously perturbing the derivative of the diffeomorphism f over the piece of orbit $\{f^j(q) \colon j = 0 \ldots N(\varepsilon) - 1\}$ by perturbations of the form

$$\begin{pmatrix} 1 & -\frac{\varepsilon}{a} \\ 0 & 1 \end{pmatrix}$$

in an appropriate base. Thus we get

$$Dg^{\pi(q)}(q) \mid_{E_q^1} \approx \lambda_p^{\pi(q)-\ell} \begin{pmatrix} a & b - \varepsilon N(\varepsilon) \\ 0 & a \end{pmatrix} \approx \lambda_p^{\pi(q)-\ell} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Hence there exists a periodic point $\tilde{q}$ such that $D\tilde{g}^{\pi(\tilde{q})}(\tilde{q}) \mid_{E_{\tilde{q}}^1}$ has simple real spectrum. This finishes the proof of the theorem.

3.4. **Proof of Theorem 4.** Let f be a C^1 diffeomorphism and Λ be a hyperbolic basic piece. Assume there are two periodic points $p, q \in \Lambda$ with robust different unstable signatures. Let $\mathcal{U}$ be a C^1-open neighborhood of f so that for any $g \in \mathcal{U}$ the analytic continuation Λ_g of Λ is well defined and let p_g and q_g denote the continuations of p and q, respectively, and $sgn^u(p_g) = sgn^u(p)$ and $sgn^u(q_g) = sgn^u(q)$. Since $\dim M = 3$ we may assume without loss of generality that $\dim(E^s) = 1$ and $\dim(E^u) = 2$.

For any $n \in \mathbb{N}$ we define $\mathcal{R}_n$ to be the set of C^1 diffeomorphisms $g \in \mathcal{U}$ such that g has n distinct periodic points with robust unstable signature equal to the one of p_g and has n distinct periodic points with robust unstable signature equal to the one of q_g.

Observe that, by hypothesis, $\mathcal{R}_1 = \mathcal{U}$ and so $\mathcal{R}_1$ is C^1-open and dense in $\mathcal{U}$. For any $n \geq 1$ the set $\mathcal{R}_n$ is clearly C^1-open. If one shows that each $\mathcal{R}_n$ is C^1-dense then the residual set $\mathcal{R} = \cap_n \mathcal{R}_n$ satisfies the statement of Theorem 4.

Assume that $\mathcal{R}_k$ is C^1-dense for any $k = 1, ..., n$ and fix $\epsilon > 0$ and any $g \in \mathcal{R}_n$. We claim that there exists $g_1 \in \mathcal{R}_{n+1}$ which is ϵ-C^1-close to g. The diffeomorphism g_1 will be obtained from g by C^1-small perturbations at two periodic points in order to obtain one more point of each robust signature. By Corollary 3.1 we know that there exists $\ell, m \in \mathbb{N}$ such that, for any periodic point x of period $\pi(x) \geq \ell$ either there exists an m-dominated splitting along the orbit of x or else for any neighborhood U of the orbit of x, there exists an $\epsilon/4$-perturbation g_1 of g in the C^1-topology, coinciding with g outside U and on the orbit of x, and for which the tangent map $(Dg_1)_x^{\pi(x)}|_{E_x^u}$ has a real eigenvalue with multiplicity two.

Assume without loss of generality that $sgn^u(p_g) = (1,1)$ and $sgn^u(q_g) = (2)$. The existence of q_g implies that the set of periodic points x of period larger that ℓ and with an m-dominated splitting along the unstable fiber is not dense in Λ_g. Indeed, if this was not the case, then q_g would have an m-dominated splitting on $E_{q_g}^u$. By the previous dichotomy it follows that q_g is accumulated by open sets without periodic points displaying an m-dominated splitting on E^u. Thus we can

pick a periodic point x distinct from the $2n$ marked periodic point for g in one of these open sets and with arbitrarily large period $\pi(x)$. Using the dichotomy there exists an $\epsilon/4$-perturbation g_1 of g in the C^1-topology, coinciding with g outside U and on the orbit of x, and for which the tangent map $(Dg_1)_x^{\pi(x)}|_{E_x^u}$ has a real eigenvalue with multiplicity two. Finally, since the periodic point x can be chosen with an arbitrarily large period we can proceed as in Subsection 3.3 and perform an $\epsilon/4$-perturbation g_2 of g_1 in the C^1-topology so that $(Dg_2)_x^{\pi(x)}|_{E_x^u}$ has a complex eigenvalue. Clearly, x has an unstable robust signature $\mathrm{sgn}^u(x) = (2)$.

If g_2 already have $n+1$ distinct periodic points with robust unstable signature equal to the unstable signature of p_g we are done. Otherwise, we are left to show that g_2 can be $\epsilon/4$-approximated in the C^1-topology by g_3 exhibiting one more distinct periodic point with robust unstable signature equal to the unstable signature of p_g. Indeed, if all but n periodic points have unstable signature (2) and since there is no dominated splitting restricted to the unstable fiber and along these orbits we can perform an $\epsilon/4$-approximated in the C^1-topology in order to obtain a distinct periodic point x with robust unstable signature equal to $\mathrm{sgn}^u(x) = (1,1)$. This completes the proof of the C^1-denseness of $\mathcal{R}_n$ in $\mathcal{U}$ and finishes the proof of the theorem.

Remark 3.3. *The previous construction of infinitely many periodic points of a certain signature has the same flavor of the construction of a generic set with infinite sinks and sources for diffeomorphisms with homoclinic tangencies: the perturbation are localized in regions where there is a lack of domination among the stable and unstable bundles.*

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